Police & Thieves: an Equilibrium Spatial Model of Crime and Surveillance*

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Abstract

Spatial heterogeneity in crime suggests that location is a key choice variable in criminal behavior. Existing spatial models represent the city as a finite set of cells: criminals choose a cell in which to commit a crime, and the police allocate a certain amount of enforcement resources across cells. We show that these models are subject to two fundamental limitations. The first one is that equilibrium outcomes depend strongly on how the city is partitioned. To address this issue, we characterize the class of discrete models whose equilibrium outcomes are invariant to the choice of spatial partition. The second shortcoming is that discrete models cannot determine the allocation of patrol units within cells or spillovers across cells, since each cell is treated in reduced form. To overcome this, we develop a continuous model in which the probability of clearing a crime depends on the distance between patrol units and the crime location. Within this framework, we characterize equilibrium crime and clearance rates and show how they depend on the clearance technology, the entropy of the spatial heterogeneity of crime attractiveness, and income inequality between criminals and victims.

JEL Codes: C72; K42; R12; C65

Keywords: spatial crime pattern; spatial policing strategy; spillover effects; apprehension probability

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1 Introduction

The expected profit from crime depends on several factors–such as the probability of finding a victim, the expected bounty, luminosity conditions, and local social dynamics. These factors are highly heterogeneous across space, even within the same city. As a result, crime incidence shows well-documented spatial variation (see, e.g., Weisburd 2015, Lee et al. 2017, Gill et al. 2017). At the same time, the spatial pattern of crime cannot be studied in isolation from police strategy, as crime and law enforcement interact in intricate ways. For example, empirical evidence shows that criminals respond to an increase in police presence in a region by committing less crime there and more elsewhere–a phenomenon known as the displacement effect, documented by Di Tella and Schargrodsky (2004) and more recently by Maheshri and Mastrobuoni (2021) and Blattman et al. (2021). Such responses may, in turn, prompt the police to reallocate their units, illustrating the mutually adaptive behavior of criminals and law enforcement.

While the empirical literature has recently made substantial progress in understanding the spatial relationship between police and crime (as illustrated in Adda et al. 2014, Collazos et al. 2021, Khanna et al. 2025), theoretical work has lagged behind. Frameworks that consider the spatial nature of criminal behavior (such as Freeman et al. 1996, Lazzati and Menichini 2016, Galiani et al. 2018) typically divide the city into geographical cells–which may, for example, represent neighborhoods–and model criminals as choosing the cell in which to commit a crime within this discrete-choice space. This modeling approach has two important implications. First, the interaction between police and criminals within each geographical cell can only be expressed in reduced form. In practice, authors choose an ad-hoc function to determine how the clearance probability depends on the number of police patrols in a cell. By doing so, any spatial heterogeneity that might influence the interplay between police and criminals within a cell is ignored. Second, because these models lack a topological notion of space, there are no spillovers of police protection between distinct cells, irrespective of their distance or size. That is, increasing police enforcement in a cell, however small, does not affect adjacent cells. This limits the applicability of discrete-choice models when cells are highly granular and spillovers are inevitable.

The contribution of this paper is twofold. First, we show that applying discrete-choice models to inherently continuous choice spaces leads to important theoretical pitfalls. In particular, we show that equilibrium outcomes in these models depend on how the city is partitioned. Second, to overcome these limitations, we develop a tractable model with an explicit topological structure–grounded in empirical findings–that accounts for the proximity between crime and police locations and provides a microfoundation for the equilibrium outcomes resulting from the interplay between these agents.

We start by outlining the main components shared by spatial models of crime–whether discrete or continuous–to provide context for our framework. These models feature two types of players: a police command and a population of criminals. Each point in the city is characterized by the expected profit from a successful crime at that location, which we refer to as crime attractiveness. The police command operates under a budget constraint and chooses how to allocate patrols across regions. Their objective is to maximize apprehensions and minimize crime, which are equivalent in our framework. The level of police enforcement in a region depends on the resources allocated there: higher police expenditures increase the probability that crimes are cleared. Criminals choose whether and where to commit crimes. They seek to maximize expected payoffs, which depend on the expected gains from criminal activity, the probability of apprehension, and an idiosyncratic cost of committing crimes.

Our starting point is to show that existing frameworks in the literature fail to satisfy a basic tautological property, which we term *coherent aggregation*. In discrete models, cities are arbitrarily partitioned into cells, each with a given payoff for successful crimes. Police protection in a cell depends on a public safety production function (PSPF), arbitrarily chosen in the models, that links police expenditures to the probability of clearing crimes in that cell. When equilibrium outcomes–such as the total crime rate and clearance rate–do not depend on the partition, we say that the PSPF satisfies coherent aggregation. We show that PSPFs used in previous studies (such as Ehrlich 1973, Merlo 2003, Lazzati and Menichini 2016, Fu and Wolpin 2017) do not satisfy this property. This is problematic because the partition of the city is an arbitrary modeling choice rather than a fundamental of the economy.

This raises the question of whether prior studies merely relied on PSPFs that fail to satisfy coherent aggregation, or this property itself is unattainable. Proposition 1 shows that some functional forms satisfy coherent aggregation, and we characterize them. We further show that imposing an additional requirement–that regions with no police enforcement have zero probability of crime clearance–substantially restricts the class of admissible functions. The remaining family is indexed by a single real parameter that captures the technology of crime clearance. By imposing these economically meaningful properties on PSPFs, we address the two sources of arbitrariness highlighted in the previous paragraph: the choice of the PSPF becomes well justified, and the choice of the partition does not affect the results, as equilibrium outcomes are invariant to it.

However, even if aggregability were not an issue, all discrete-choice models share a common limitation: they ignore potential spatial spillover effects arising from police patrols that provide protection beyond a single cell. In a discrete framework, a cell without assigned patrols receives no protection, even if neighboring cells are heavily patrolled, which is particularly unrealistic when cells are small. Therefore, discrete models face an inherent trade-off. When cells are large, spillovers are less significant, but police dispatching and crime rates can only be characterized in aggregated terms over broad areas. Achieving a finer spatial description of equilibrium outcomes requires using smaller cells; Nonetheless, this introduces spillover effects between neighboring cells that discrete models cannot adequately capture.

Our second main contribution addresses this limitation by considering a topological formulation of the crime model. In this framework of crime and surveillance, the probability that a crime is cleared depends on both the number and the proximity of patrols to the crime location. Because the interaction between police and criminals is microfounded, the model naturally satisfies the key properties, among them coherent aggregation. In particular, if the city is partitioned and the PSPF derived in Proposition 1 is applied, aggregate equilibrium outcomes such as crime rate and the proportion of cleared crimes coincide in the corresponding discrete model and in our continuous framework. Moreover, because the police command problem is formulated directly as the allocation of units, the model has the additional advantage of allowing us to characterize equilibrium patrol locations–something discrete models cannot achieve, as we illustrate with numerical examples in Subsection 4.3.

The full characterization of the equilibrium is provided in Proposition 2. Our tractable framework allows a rich interpretation of the determinants of the aggregate clearance rate, the crime rate, and the spatial distributions of crime and police patrols. We show how the spatial preferences of criminals affect the clearance rate. When crime attractiveness is more concentrated (i.e., less homogeneous) in some regions, the police force has more information on where crimes are more likely to be committed, enabling it to clear more crimes. In particular, the worst-case scenario for the police arises when attractiveness is uniform across the jurisdiction.

The level of crime in our model depends on three factors. First, the quality of the clearance technology and the overall level of police protection, which together determine the expected losses for agents considering engaging in criminal activity. Second, the relative entropy between the spatial distribution of attractiveness and the uniform distribution, which affects crime levels by informing the police where crimes are more likely to occur. Third, income inequality between criminals and victims, as a greater amount of lootable goods held by potential victims—without a proportional increase in criminals' income—raises the crime rate. This aligns with several empirical studies, including Enamorado et al. (2016) and Buonanno and Vargas (2019), that find a positive association between crime and inequality.

The rest of the paper is organized as follows. The next section lays out the general problem, notation, and properties used in both discrete and continuous models. In Section 3, we present the generic discrete framework, show how typical models used in the literature fail to satisfy coherent aggregation, and derive the family of discrete models that respect this property. Section 4 presents the continuous model and its equilibrium, and Section 5 concludes the paper.

2 The General Problem

We aim to develop a spatial model that captures the interaction between police and criminals. This section summarizes the components common to both discrete- and continuous-choice crime models.[1] The police command operates under a budget constraint and must decide how to allocate patrols across different regions. We assume that their objective is to maximize apprehensions and minimize crime, which, in our framework, are equivalent.[2] The level of police enforcement in a region depends on the resources allocated by the police command. Increasing police expenditures raises the probability of clearing crimes in that region, as supported by, for example, Levitt (2002) and Evans and Owens (2007). Criminals choose where to commit crimes to maximize their payoffs, taking into account both the expected gains from criminal activity and the probability of apprehension. We require our model to satisfy two criteria: (i) it must fulfill certain desirable properties–including coherent aggregation–, and (ii) it must inform us about the actual allocation of police units and crime occurrences across the police jurisdiction.

[1]Additional assumptions used exclusively in the discrete- and continuous-choice models are introduced in Section 3 and Subsection 4.1 respectively.

[2]Corollary 3 in Subsection 4.2 provides a formal proof of this equivalence for the continuous model we propose. We adopt the maximization of apprehensions as the police objective to streamline the exposition.

2.1 Notation, Fundamentals, and Equilibrium Concept

To fix notation, let $W \subset \mathbb{R}^2$ be a compact set representing a police force's jurisdiction. We refer to an arbitrary two-dimensional subset $W' \subset W$ as a *region*, and its area is denoted by $|W'|$. The set of positive continuous functions on W is denoted by $\mathcal{C}_{++}$. For any non-negative integrable function v on W such that $\int_W v(x)\mathrm{d}x > 0$, we define f_v as the normalization of v, that is, $f_v := v / \int_W v(x)\mathrm{d}x$. We define δ as the Dirac delta function, so that $\int_{W'} \delta(x - x_0)\mathrm{d}x = 1$ if $x_0 \in W'$ and 0 otherwise.

We next describe the fundamentals of this game. Law enforcement determines that the police command operates under a budget Λ and applies a punishment b to a criminal when a crime is cleared. We consider a pool of N risk-neutral potential criminals who contemplate the possibility of committing crimes. Each criminal has an opportunity cost drawn from a cumulative distribution function (CDF) F_κ to engage in criminal activity. By incurring this cost, an agent becomes a criminal and selects a location $x \in W$ in which to commit the crime.

We also define the criminals' attractiveness function $a \in \mathcal{C}_{++}$. This function is interpreted as follows: if a criminal can choose a point $x \in W$ to commit a crime (as in a continuous-choice model), $a(x)$ represents the expected monetary gain if the crime is not cleared; if the criminal can choose a region $W' \subset W$ to commit a crime (as in a discrete-choice model), the expected monetary gain if the crime is not cleared is the average of a over W'.[3] Figure 1 illustrates an example of the attractiveness function in a city $W = [0, 1] \times [0, 1]$. The colors in the figure represent the attractiveness at each point within the city. We can see two points where the attractiveness is maximal.[4]

Motivated by empirical evidence suggesting that criminals and police learn about each other's preferences through repeated interactions (see, e.g., Knowles et al. 2001), we adopt the Nash equilibrium with rational expectations as our solution concept. Thus,

[3] Despite studies on the statistical relation between neighborhood characteristics and crime patterns dating back to the 19th century (see Guerry 1833 and Quetelet 1842), the determinants of crime distribution are disputed in the criminology literature. Therefore, we deliberately refrain from providing a microfoundation for it–we only require that the criminals' preferences exist and are well-defined. For example, while some empirical papers find that the density of potential victims attracts crime (Brantingham and Brantingham 1995, Harries 2006, Cabrera-Barona et al. 2019), others find that crime is attracted to a weaker social cohesion (Shaw and McKay 1942, Sampson and Groves 1989, Bernasco and Block 2009), and yet others find that criminals prefer to commit crimes in regions similar to their neighborhood (Chamberlain and Boggess 2016).

[4] The attractiveness level at $(x, y) \in W$ is given by

$$z(x, y) = \exp\left(-\alpha_0\left((x - x_0)^2 + \alpha_1(y - y_0)^2\right)\right) + \exp\left(-\alpha_2\left((x - x_1)^2 + \alpha_3(y - y_0)^2\right)\right).$$

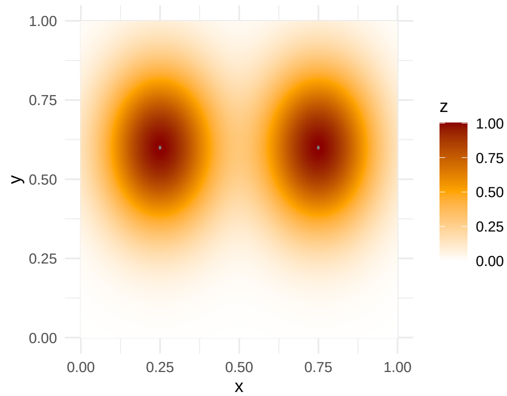


Figure 1: Example of attractiveness level within $W = [0, 1] \times [0, 1]$

although the game is formally static, the Nash equilibrium captures an implicit dynamic interaction between criminals and the police.[5] Variables with a superscript $\star$ denote equilibrium values.

2.2 Desirable Properties of Models

We now introduce definitions that formalize two desirable properties that crime models should satisfy. We let $\{W_i\}_{i \in I}$, $I \subseteq \mathbb{N}$, be a collection of disjoint sets in W with $|W_i| > 0$, and we define $a(W_i) = \frac{1}{|W_i|} \int_{W_i} a(x) \mathrm{d}x$ as the average attractiveness in W_i. If we set $a(\{x\})$ as the limit of sets with positive areas converging to x, we have $a(\{x\}) = a(x)$.[6]

Regarding equilibrium variables, for any $W' \subset W$ with $|W'| > 0$, let $M^\star_{W'}$ be the expected number of crimes, $p^\star_{W'}$ the expected proportion of crimes that are cleared, and $\Lambda^\star_{W'}$ the police expenditures in W'. The aggregate clearance rate (ACR) of $\bigcup_{i \in I} W_i$ under the partition $\{W_i\}_{i \in I}$, denoted $p^\star_{\{W_i\}_{i \in I}}$, is defined as

$$p^\star_{\{W_i\}_{i \in I}} := \frac{\sum_{i \in I} M^\star_{W_i} p^\star_{W_i}}{\sum_{i \in I} M^\star_{W_i}}.$$

Note that when the partition consists of a single set, $\{W_i\}_{i \in I} = \{W_i\}$ and $p^\star_{\{W_i\}} = p^\star_{W_i}$.

The first property we require from models is the coherent aggregation principle,

[5] The way learning leads to a Nash equilibrium in contexts similar to ours is discussed in Kalai and Lehrer (1991).

[6] Formally, if $(W'_n)_n$ is a sequence of measurable sets with $|W'_n| > 0$ for each n converging to a point $x \in W$ with respect to the Hausdorff distance, and we define $a(\{x\}) := \lim_{n \to \infty} a(W'_n)$, then $a(\{x\}) = \int_W a(x') \delta(x' - x) \mathrm{d}x' = a(x)$.

which states that equilibrium outcomes cannot depend arbitrarily on the partition of the police jurisdiction W.

Coherent Aggregation Principle. *We say that a model satisfies coherent aggregation when the following three conditions hold:*

(i) The equilibrium crime rate in W does not depend on the way the set is partitioned;

(ii) The ACR of a set $\bigcup_{i \in I} W_i$ does not depend on how the regions are aggregated, so

$$p^{\star}_{\{W_i\}_{i \in I}} = p^{\star}_{\{\bigcup_{i \in I} W_i\}};$$

(iii) Police enforcement expenditures are additive across regions, hence

$$\sum_{i \in I} \Lambda^{\star}_{W_i} = \Lambda^{\star}_{\bigcup_{i \in I} W_i}.$$

The second property we impose on models of crime is that if the surface density of enforcement around a point–defined as the ratio of police expenditures to area in a region–converges to zero, then the ACR around that point must also converge to zero. This captures the idea that, in the absence of police enforcement in the vicinity of a point, no crimes committed there can be cleared. We formalize this requirement as follows.

No-Enforcement, No-Clearance Principle. *For any sequence $\{W'_i\}_i$ of subsets of W such that $W'_i \to \{x\}$ (under the Hausdorff distance) and any sequence of expenditures $\{\Lambda^{\star}_{W'_i}\}_i$ satisfying $\lim_{i \to \infty} \Lambda^{\star}_{W'_i}/|W'_i| = 0$, it follows that $\lim_{i \to \infty} p^{\star}_{W'_i} = 0$.*

We call this the NENC principle, and any model that respects both principles is referred to as a *consistent model.*

3 A Discrete Framework

In this section, we solve a generic version of the choice models used in the literature to show that these frameworks do not allow for coherent aggregation in their most general formulation. Consider a partition of W, given by $\{W_i\}_{i \in I}$, where $I \subseteq \mathbb{N}$. The police can allocate a portion of the budget to each region, and each criminal decides whether and where to commit a crime. If a crime in W_i is not cleared, the criminal earns a profit equal to the average attractiveness of W_i, given by $a(W_i)$.

The critical modeling decision is how to set the public safety production function (PSPF). That is, the probability of clearing a crime in each region W_i as a function of the expenditures on police in W_i, denoted by Λ_i. We impose that this function depends only on the fundamentals of W_i (the shape and the function a inside it), that it is bounded in $[0, 1]$, strictly increasing, and strictly concave in Λ_i. We denote the clearance probability of region W_i as $\pi_{W_i}(\Lambda_i)$ for each $\Lambda_i \in \mathbb{R}_+$. Observe that $p^\star_{W_i} = \pi_{W_i}(\Lambda^\star_{W_i})$.

3.1 Solving the Equilibrium

We solve the equilibrium assuming that the equilibrium expenditures on police are strictly positive for each W_i. One way to interpret this assumption is that we focus only on the regions of the jurisdiction that are attractive enough to induce crime and, therefore, require enforcement. Alternatively, this assumption can be seen as a consequence of having a sufficiently high budget Λ. Finally, we avoid excessive formalism, as solving this model does not yield insights beyond those already present in the literature.

We start with the criminals. The average profit of a successful crime in region W_i is $a(W_i)$, thus, the expected profit from committing a crime in region W_i in equilibrium is

$$a(W_i)(1 - \pi_{W_i}(\Lambda^\star_i)) - b\ \pi_{W_i}(\Lambda^\star_i).$$

In equilibrium, criminals must be indifferent as to which region they commit a crime. This is a well-known result: if there were a region W_i preferred over all others, then all crime would be concentrated there. This would prompt the police command to allocate all the resources to that region. Facing high police enforcement, W_i would no longer yield high expected profits for criminals, contradicting the assumption that W_i is the preferred region.

Therefore, the equilibrium value of committing a crime in each region is constant, which we denote as $V^\star$. Note that this value might, in principle, depend on the partition of W and on the functionals $(\pi_{W_i})_{i \in I}$. A necessary condition for equilibrium is then

$$(a(W_i) + b)\,(1 - \pi_{W_i}(\Lambda^\star_{W_i})) = V^\star + b.$$

By inverting the π_{W_i} functions, we can determine the optimal equilibrium expenditure levels for each region, given by

$$\Lambda^{\star}_{W_i} = \pi^{-1}_{W_i}(1 - (V^{*} + b)/(a(W_i) + b)). \tag{1}$$

Now, we can analyze the problem of the police command. Recall that $M^{\star}_i$ is the equilibrium number of crimes in region W_i. The police maximizes the total expected apprehensions, subject to the budget constraint, which is given by

$$\max_{\Lambda_1, \Lambda_2, \ldots, \Lambda_n} \sum_{i \in I} M^{\star}_i \pi_{W_i}(\Lambda_i) \quad \text{s.t.} \quad \sum_{i \in I} \Lambda_i \leq \Lambda$$

If we attach the Lagrangian multiplier ζ to the budget constraint, the first-order condition of the problem implies that

$$M^{\star}_i = \zeta / \pi'_i(\Lambda^{\star}_{W_i}) \tag{2}$$

whenever $M^{\star}_i > 0$. Since we assume that there is police enforcement in each region in equilibrium, $M^{\star}_i > 0$ for each $i \in I$.

Thus, we can write the ACR using the partition $\{W_i\}i \in I$ as

$$p^{\star}_{\{W_i\}_{i \in I}} = \frac{\sum_{i \in I} M^{\star}_i \pi_i(\Lambda^{\star}_i)}{\sum_{i \in I} M^{\star}_i}.$$

Using equations (1) and (2), we can write the ACR down as a function of the fundamentals and $V^{\star}$:

$$p^{\star}_{\{W_i\}_{i \in I}} = 1 - (V^{\star} + b) \frac{\sum_{i \in I} \frac{\pi^{-1'}_{W_i}(1 - (V^{\star} + b)/(a(W_i) + b)}{a(W_i) + b}}{\sum_{i \in I} \pi^{-1'}_{W_i}(1 - (V^{\star} + b)/(a(W_i) + b))}.$$

Finally, $V^{\star}$ is obtained using the budget constraint:

$$\sum_{i \in I} \pi^{-1}_{W_i}(1 - (V^{*} + b)/(a(W_i) + b)) = \sum_{i \in I} \Lambda^{\star}_{W_i} = \Lambda.$$

The model is, thus, completely characterized.

3.2 PSPFs Used in the Literature

In the literature, it is common for some articles to specify a given function π_{W_i}. For example, Merlo (2003) considers the following specification:

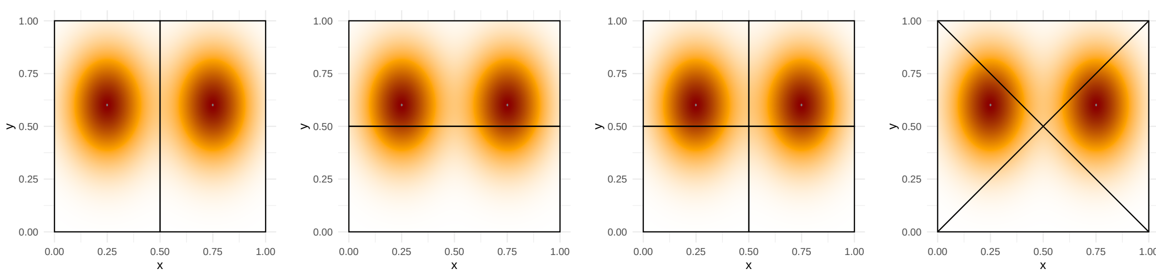
$$\pi_W(\Lambda) = \begin{cases} 0 & \text{if } \Lambda \leq 1 \\ 1 - \xi \Lambda^{-\alpha} & \text{if } \Lambda > 1, \end{cases}$$

with $\xi > 0$ and $\alpha \in (0, 1)$. Fu and Wolpin (2017) specify the function

$$\pi_{W_i}(\Lambda_i) = \exp\left(-\frac{\gamma}{\Lambda_i}\right),$$

with $\gamma > 0$, when the level of crime has no impact on the clearance probability.

We use these two specific functions and the attractiveness pattern from Figure 1 to illustrate how this approach fails to satisfy the coherent aggregation principle, as both the level of crime and the ACR implied by these models depend on how the city is subdivided. To demonstrate this, we consider four different ways of partitioning our unitary city into regions, depicted in Figure 2. Splits A and B divide the city into two areas of equal size, each being a rectangle measuring 1×0.5, yielding an area of 0.5 units. Splits C and D subdivide the city into four units of equal area. In Split C, we obtain four squares with side length 0.5, while in Split D, the regions are isosceles triangles of area 0.25.



(a) Split A (b) Split B (c) Split C (d) Split D

Figure 2: Example of partitions of a city

We solve numerically for the equilibrium of this economy under different splits and different functional forms for the technology of clearing crimes used by Merlo and by Fu and Wolpin. We set $\Lambda = 100$ (which induces interior solutions in all cases), $b = 0.1$, $N = 1$, F_κ as the uniform distribution on $[0, 1]$, and values of α, γ, and ξ for the two different clearance functions so as to yield an ACR of 0.7 in the split A case.[7] As the areas in splits A and B differ from those in splits C and D, we compute Λ_i as the expenditure per unit of area.

Table 1 shows the results. We can see that with Split A and the Merlo (2003) and Fu and Wolpin (2017) functions, we have an ACR of 0.7 (by construction) and a level of crime of 0.1152. If we solve for optimality under split B, which also has two subregions of the same size as split A, we would have a different result with an ACR of 0.7336 and a criminal rate of 0.1282 in Merlo's model and an ACR of 0.7249 and a criminal rate of 0.1151 in Fu and Wolpin's model. The table shows that these results depend on the number of splits as well as on the specific geometric format of these splits. In other words, the same discrete model of crime can yield different results depending on how the city is arbitrarily partitioned.

Table 1: Average Clearance Rate and Crime Rate by technology and split

Clearance Technology	Split	# Units	ACR	LC
Merlo (2003)	Split A	2	0.7000	0.1152
	Split B	2	0.7336	0.1282
	Split C	4	0.7336	0.1282
	Split D	4	0.7281	0.1301
Fu and Wolpin (2017)	Split A	2	0.7000	0.1152
	Split B	2	0.7249	0.1151
	Split C	4	0.7249	0.1151
	Split D	4	0.7328	0.1148
Consistent Model	Split A	2	0.7000	0.1152
	Split B	2	0.7000	0.1152
	Split C	4	0.7000	0.1152
	Split D	4	0.7000	0.1152

3.3 A Consistent Model

The lack of coherent aggregation in the models of Merlo (2003) and Fu and Wolpin (2017) is common to all other discrete-choice models of crime. A natural question,

[7]More specifically, for Merlo (2003) we have $\pi_M = 1 - \xi(\Lambda + 1)^{-\alpha}$, with $\xi = 1$ and $\alpha = 0.26$. For Fu and Wolpin (2017) we have $\pi_{FW}(\Lambda) = \exp\left(\frac{-\gamma}{\Lambda}\right)$, with $\gamma = 35.67$.

therefore, is whether a consistent model exists. That is, is there a functional form for the relationship between the clearance rate in a region and the expenditures on police in that region that allows for coherent aggregation and respects the NENC principle? We show that such a functional form exists, that it is unique up to a scalar parameter, and we characterize it in the following proposition.

Proposition 1. *Let $W' \subseteq W$ be a measurable set with $|W'| > 0$. The only family of functional forms for the clearance probability that allows for coherent aggregation is*

$$\pi_{W'}(\Lambda') = 1 - e^{-\frac{\Lambda'\theta}{|W'|}g(W')} \tag{3}$$

for any $\Lambda' \geq 0$, where θ is a positive constant and $g : 2^W \to \mathbb{R}_{++}$ is given by

$$g(W') = \frac{e^{\frac{1}{|W'|}\int_{W'} h(x)dx}}{a(W') + b} \tag{4}$$

for any integrable function $h : W \to \mathbb{R}$. If, moreover, the NENC principle is respected, then h is unique and given by $h = \ln(a + b)$ and in a consistent model, we must have

$$g(W') = \frac{e^{\frac{1}{|W'|}\int_{W'} \ln(a(x)+b)dx}}{a(W') + b}.$$

Equations (3) and (4) state that the family of functions respecting the coherent aggregation principle is as large as the family of signed measures on W. For most of these measures, there will be at least one point in W such that the probability of clearing crimes around this point, when the ratio of police expenditures to area goes to zero, is not zero (it can be positive or even negative). The only g that ensures the NENC principle is the one mapping $W' \to \int_{W'} \ln(a(x) + b)\mathrm{d}x$. That is, if we want a consistent model, we must have

$$\pi_{W'}(\Lambda') = 1 - e^{-\frac{\Lambda'\theta}{|W'|}\frac{e^{\frac{1}{|W'|}\int_{W'} \ln(a(x)+b)\mathrm{d}x}}{a(W') + b}} \tag{5}$$

for each $\Lambda' \geq 0$. This is a one-parameter function, indexed by θ, that depends on three aggregate quantities over W': the area of W', given by $\int_{W'} 1\,\mathrm{d}x$; the average attractiveness in W', given by $\int_{W'} a(x)\,\mathrm{d}x$; and a term related to the relative entropy of $a + b$ with respect to the uniform distribution, given by $\int_{W'} \ln(a(x) + b)\,\mathrm{d}x$. Table 1 shows the results for this consistent model. We choose θ so as to yield an ACR of 0.7

under split A.[8] As expected, the ACR and criminal rates are the same irrespective of how the city is partitioned.

3.4 A Drawback of Discrete Models

The functional form for the clearance probability described by equation (5) allows for a consistent model of crime. Besides, it displays other desirable properties. The clearance probability is strictly increasing and strictly concave in police enforcement, it is continuous and differentiable, and computationally simple.

However, discrete-choice frameworks—even consistent ones—are unable to determine the equilibrium patrol deployment within a cell. A natural approach to deal with this problem is to refine the spatial partition by making cells arbitrarily small, but this creates an additional issue. When cells are small, patrols assigned to a given cell are likely to provide protection to neighboring cells. By treating each cell in isolation from the others, discrete models thus fail to capture spillover effects across regions. Therefore, discrete models face an inherent tradeoff. If the jurisdiction is partitioned into a small number of large cells, the resulting outcomes are silent on what happens inside each cell, providing an incomplete characterization. If the partition generates a large number of small cells, it improves the characterization of actual police deployment and crime rates, but it abstracts away from spillovers and how the small cells interact.

Analytically, in the absence of spillover effects, the equilibrium spatial distribution of police patrols depends pointwise on local crime attractiveness. That is, in discrete-choice models, if the attractiveness at $x \in W$ and $x' \in W$ satisfies $a(x) = a(x')$, then expenditures on police at x and x' (and, consequently, the number of patrols around these points) are necessarily identical, regardless of the attractiveness in the surrounding areas of x and x'. The next section shows how a continuous model departs from this pointwise dependence and allows the spatial distribution of patrols to depend on the entire attractiveness function.

4 A Continuous Framework

In this section, we develop a continuous-choice model in which the level of enforcement depends on the actual distance between the location of a crime and police patrols. This approach allows us to address several issues simultaneously.

[8] Specifically, $\theta = 0.0096$.

First, our model relies on empirically grounded assumptions. Evidence suggests that reducing the distance between patrol units and the crime location increases the likelihood of clearing the crime (see Weisburd 2021, Mastrobuoni 2019, and Blanes i Vidal and Kirchmaier 2018). As Blanes i Vidal and Kirchmaier emphasize, while patrol officers rarely apprehend suspects on site, closer proximity improves the preservation of material evidence and enhances the probability of identifying witnesses with relevant information.

Second, the model provides a microfoundation for the clearance mechanism, rather than modeling clearance probabilities in reduced form, as is common in the literature. By considering the maximal level of spatial disaggregation of W–with all points admissible as both crime locations and sites of police protection–our model is consistent.

Third, the model delivers the equilibrium distribution of crime and patrols at each point, not merely their aggregate levels, as discussed in Subsection 3.4. Also, in our model, depending on the cover radius of each patrol, the density of units may not align with areas of higher criminal activity–a feature discrete-choice models cannot capture.

Finally, the model is both analytical and flexible, enabling a clearer understanding of the strategic interaction between criminals and the police, as well as the policy implications.

Before presenting the formal model, we highlight key differences relative to discrete-choice frameworks. In our continuous-choice model, the police command determines the probability of patrol presence at each point of W as perceived by criminals. The probability of clearing a crime at location x depends on the proximity of patrols, so each point in W is associated with a distinct clearance probability. Given that each point in W is characterized by a probability of crime occurrence and a probability of clearance, the police's total payoff–the utility of the police command–is obtained by integrating the product of these two quantities over W.

As before, each criminal decides whether to enter, but now chooses a point $x \in W$ at which to commit a crime. If the crime occurs at x, the criminal receives a payoff $a(x)$ if it is not cleared and incurs a penalty b otherwise. The probability of clearance at x depends on the distribution of police patrols, and each potential criminal maximizes their expected payoff.

4.1 Model Set-up

In what follows, we formally describe the main components of the model.

The Clearance Mechanism

To capture the dependence of the clearance probability on the proximity of patrols, we introduce a *crime-detecting kernel* K. If x denotes the location of a crime and y the location of a police patrol, the probability that the patrol detects the crime is given by $K(\|x-y\|)$. The function K is decreasing and satisfies $K(t) = 0$ for $t > r$. We assume that r is small relative to the overall dimensions of the city or region where crimes may occur, so that border effects can be neglected.

The simplest kernel is the uniform one, where $K = \beta \mathbb{1}_{\{(\cdot) \leq r\}}$ for $\beta \in [0,1]$. In this case, a patrol detects a crime with probability β if the distance between the crime and the patrol is less than or equal to r, and with probability 0 if the distance exceeds r. Figure 3 illustrates three examples of different crime-detecting kernels.

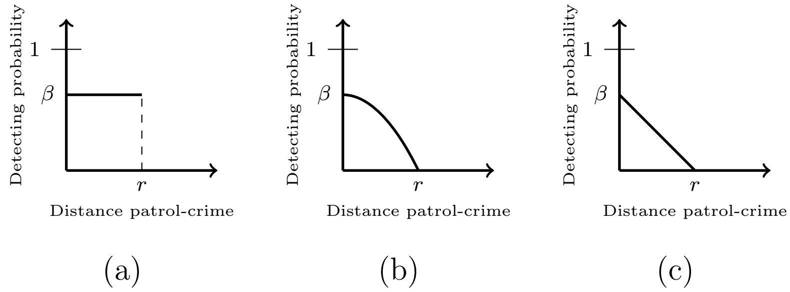


Figure 3: Different crime-detecting kernels. Panel (a) presents a uniform kernel, panel (b) a parabolic one, and panel (c) a linear one. In all cases, the maximum detection probability is β (potentially equal to one) when the crime occurs at the location of the police patrol, and this probability decreases to 0 as the distance between the patrol and the crime exceeds r.

A crime occurring at location x is cleared if at least one police patrol detects it.[9] Consequently, if, for example, four police patrols are near x, the probability of clearing the crime is typically higher than if only a single patrol were present. This reflects the idea that areas with greater police presence have a higher likelihood of crime clearance.

Therefore, if there are N_p patrols and the set of points $\{y_1, y_2, \ldots, y_{N_p}\}$ represents the locations of police patrols, the probability that a crime at $x \in W$ is cleared is

$$1 - \prod_{j=1}^{N_p} \left(1 - K(\|x - y_j\|)\right).$$

[9]The key idea is that proximity to patrols increases the probability of clearance. One could also assume that when the distance between a patrol and a crime exceeds r, there remains a positive constant probability of clearance (we assume this constant is zero). Incorporating this assumption does not affect our main results and would align more closely with Blanes i Vidal and Kirchmaier (2018) (see Figure 4 in their paper).

Figure 4 shows the clearance probabilities as a function of the locations of crimes and police patrols with a uniform crime-detecting kernel. Darker areas correspond to higher clearance probabilities.

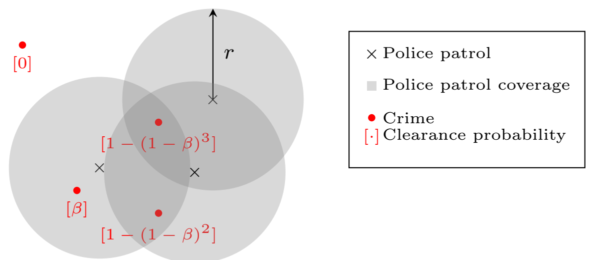


Figure 4: Clearance probability as a function of the locations of patrols and crimes within W under a uniform crime-detecting kernel. A patrol located within a radius r of a crime detects it with probability β.

Informational Structure and Point Processes

Since the police command does not know the exact locations of crimes in advance, crime locations are random from the police perspective, and we model them as a point process Φ. A point process can be viewed as a random element, analogous to a random variable, but taking values in the space of locally finite point configurations. While the probability law of a real-valued random variable Z determines, for example, the probability that Z falls within the interval $[0, 1]$, the probability law of a point process Φ specifies the probability that, for instance, no more than three points of Φ fall within a given region $A \subset W$. The distribution of Φ is determined endogenously by the equilibrium, and we do not, in principle, rule out the theoretical–albeit unrealistic–possibility that Φ follows a degenerate distribution, resulting in a deterministic set of crime locations.

One of the properties of a point process is its *intensity*. We denote the intensity of Φ by μ, which characterizes the superficial density of crimes. That is, while Φ represents the record of exact locations of crimes, μ can be seen as the heat map of crimes, indicating the areas where crimes are more likely to occur. More precisely, the quantity $\mu(x)\,\Delta x$ can be interpreted both as the probability of a crime occurring in a small area Δx around x and as the expected number of crimes around x. Intuitively, the expected number of crimes occurring around the points $\{x', x''\}$ is $\sum_{x\in\{x',x''\}} \mu(x)\,\Delta x$. Similarly, the expected number of crimes occurring in a set A, $\mathbb{E}\Big[\sum_{x\in\Phi} \mathbb{1}_{\{x\in A\}}\Big]$, is given by $\sum_{x\in A}\mu(x)\,\mathrm{d}x = \int_A \mu(x)\,\mathrm{d}x$, and $\int_W \mu(x)\,\mathrm{d}x$ is the total expected number of crimes.

This relation formally holds and, more generally, if $f : W \to \mathbb{R}_+$ is a measurable function, m is the number of criminals, and $\Phi = \{x_1, x_2, \ldots, x_m\}$, we have that

$$\mathbb{E}[f(x_1) + f(x_2) + \cdots + f(x_m)] = \mathbb{E}\Big[\sum_{x\in\Phi} f(x)\Big] = \int_{\mathbb{R}^2} f(x)\mu(x)\mathrm{d}x,$$

and we refer to this result as the *intensity property* (see Weil 2007).

Analogously, as long as potential criminals do not know the exact locations of police patrols in equilibrium, the set of patrol locations appears random from the criminals' viewpoint. Thus, we model patrol deployment as a point process Ψ, respecting properties that will be defined later. We denote by λ the intensity of Ψ, and $\lambda(x)\,\Delta x$ represents the probability of finding a patrol in a small region around x with area Δx. The randomness of Ψ does not imply that the police command is unaware of patrol locations; rather, it reflects the strategic intention of conducting preventive patrols, making the presence of police appear unpredictable to potential criminals.

Because the police command deploys patrols before knowing the locations of crimes, and criminals choose crime locations without knowing the locations of patrols, Ψ and Φ are independent point processes.[10] Here and in the following, *independent* refers to probabilistic independence, which does not prohibit, for example, the police from favoring patrol deployment in locations with a high intensity of crimes.

Since the theory of point processes is not standard in Economics, we illustrate a few examples that could characterize Φ, along with their associated distribution laws and intensity measures.

Example 1. Suppose that there is exactly one crime uniformly distributed over W (this is known as a Binomial point process). Then, it is easy to see that the probability law of Φ is described as: for any $A \subset W$, $\Pr((\#\text{ crimes in } A) = 0) = 1 - |A|/|W|$. In addition, the intensity of Φ in this case is given by $1/|W| \times \mathbb{1}_{\{x\in W\}}$. That is, the probability that the crime is around $x \in W$ is $1/|W|\mathrm{d}x$. Observe that the expected number of crimes in W is $\int_{\mathbb{R}^2} 1/|W| \times \mathbb{1}_{\{x\in W\}}\mathrm{d}x = 1$. $\square$

Example 2. Suppose that the average number of crimes in W follows a Poisson distribution with mean 1, and the location of each crime event is independently and

[10]Under this assumption, we abstract from the possibility of direct deterrence, whereby a criminal refrains from committing a crime upon observing a police patrol.

uniformly distributed over W (this is known as a Poisson point process). Then it is possible to show that the probability law of Φ is fully described as: for any $A \subset W$, $\Pr((\#\text{ crimes in } A) = 0) = e^{-|A|/|W|}$. As in the previous example, the intensity of Φ is $1/|W| \times \mathbb{1}_{\{x \in W\}}$ (i.e., the intensity alone does not completely characterize a point process).[11] The expected number of crimes in W is $\int_{\mathbb{R}^2} 1/|W| \times \mathbb{1}_{\{x \in W\}} \mathrm{d}x = 1$. $\square$

Example 3. Suppose that Φ is deterministic and there are exactly three crimes in W located at x_1, x_2, and x_3, respectively. In this case, $\Pr(\Psi = \{x_1, x_2, x_3\}) = 1$. The intensity is given by $\delta(x - x_1) + \delta(x - x_2) + \delta(x - x_3)$. The expected number of crimes in W is $\int_{\mathbb{R}^2} (\delta(x - x_1) + \delta(x - x_2) + \delta(x - x_3))\mathbb{1}_{\{x \in W\}} \mathrm{d}x = 3$ since $x_1, x_2, x_3 \in W$. $\square$

The Police Command

We assume the following two properties for the locations of police patrols, represented by Ψ. First, the number of patrols available to the police command is modeled as a Poisson random variable with exogenous mean Λ, which can be interpreted as a function of police expenditures determined by political agents. The uncertainty in this number reflects the bureaucratic, legal, and logistical constraints that affect the actual deployment of police units. In addition, since the standard deviation of the patrol count, $\sqrt{\Lambda}$, is small relative to its mean when Λ is large, the number of patrols can be approximated by Λ with a small relative deviation. Therefore, the results are statistically similar to those obtained under a fixed (deterministic) number of patrols.

Second, conditional on the number of patrols, the police command deploys each unit so that potential criminals perceive them as independently and identically distributed over W. We assume that this distribution is continuous. Although this places a restriction on the strategy set available to the police command, this environment still supports a wide range of policing strategies. Indeed, provided the budget constraint is respected, it allows the police command to achieve any desired level of local enforcement ex-ante (i.e., any arbitrary value of intensity at some point $x \in W$). In addition, from the police command's perspective, the precise *ex post* locations of police patrols are irrelevant, as it aims to maximize its *expected* payoff.

The two properties of Ψ, taken together, imply that the set of patrol locations

[11] To grasp the intuition, observe that when $|A|$ is small we have that $\Pr((\#\text{ crimes in } A) > 1) \ll \Pr((\#\text{ crimes in } A) = 1)$ because it is unlikely that two points fall in A. This means that $\Pr((\#\text{ crimes in } A) = 1) \approx \Pr((\#\text{ crimes in } A) > 0) = 1 - e^{-|A|/|W|} \approx |A|/|W|$. When $A \in W$ tends to a point with area $\mathrm{d}x$, we have that $\Pr((\#\text{ crimes in } A) = 1) = 1/|W|\mathrm{d}x$.

follows a Poisson point process (PPP). Moreover, because the distribution of each patrol is continuous, the intensity measure of the police is in $\mathcal{C}_+$. A notable characterizing property of the PPP is that the number of points falling in disjoint regions of W is independent and Poisson-distributed. Loosely speaking, what happens at a location in W (for example, whether or not there is a patrol) does not affect what happens in other regions of W. This confers analytical tractability, ease of implementation, and good statistical behavior to the PPPs.[12] The properties of PPPs we use in this paper are detailed in the Appendix A.

Now we characterize the objective function of the police command. Each cleared crime increases the police command's payoff by one unit. Therefore, the expected utility of the police command is given by

$$U^{\mathrm{pol}}(\lambda, \mu) := \mathbb{E}\Big[\sum_{x\in\Phi}\Big(1 - \prod_{y\in\Psi}(1 - K(\|x-y\|))\Big)\Big],$$

where the intensities of Φ (crime locations) and Ψ (patrol deployment) are given, respectively by μ (crime density) and λ (police density). By using the properties of point processes, we can show that

$$U^{\mathrm{pol}}(\lambda, \mu) = \int_W \mu(x)\Big(1 - e^{-\int_W K(\|x-y\|)\lambda(y)\mathrm{d}y}\Big)\mathrm{d}x. \tag{6}$$

Appendix B details the steps to derive (6). The term $\mu(x)\mathrm{d}x$ is the maximum expected payoff the police command can receive for protecting the point x, while the term in parenthesis represents the expected proportion of crimes around x that are cleared for a given patrolling intensity λ. Therefore, the police command seeks to maximize the right-hand side of (6) over $\lambda \in \mathcal{C}_+$, subject to the constraint $\int_W \lambda(x)\mathrm{d}x \leq \Lambda$. Note that no assumptions about Φ or its intensity μ are required to obtain (6).

[12]PPPs also have nice dynamic properties. A PPP can be seen as the equilibrium state of a Brownian-based diffusion. Consider an initial configuration of patrols distributed as a PPP with intensity λ. Assume that each point is displaced according to a Brownian motion with a drift $\nabla \ln(\lambda)$, where ∇ is the gradient operator. Ignoring border effects, as the patrols move, their distribution remains that of a PPP, preserving the initial intensity λ. This dynamic process could mirror some randomness in the movement of patrols, with the drift ensuring that the patrols provide the protection intensity chosen by the police command. The observed pattern of patrols is interpreted as a static snapshot of the dynamic process described above. See Stoyan et al. (1987) for more details on PPPs.

Criminals

As in the discrete-choice model, the number of potential criminals is N, each agent faces an idiosyncratic opportunity cost of committing a crime, drawn from a distribution with CDF F_κ, and the expected bounty from a successful crime at x is $a(x)$.[13] In addition to the entry decision, each potential criminal chooses a pure or mixed strategy over locations in W at which to commit a crime. We denote by $\mathcal{S}$ the space of distributions over W.

Now we write the utility of criminals as a function of the police patrol intensity λ and an arbitrary distribution s over W, representing the criminal's strategy. Let X be a random location drawn from s. Then, the expected payoff from committing a crime under this strategy is given by

$$U^{\mathrm{crim}}(\lambda, s) := \mathbb{E}\Big[a(X) \prod_{y \in \Psi} (1 - K(\|X - y\|)) - b\Big(1 - \prod_{y \in \Psi} (1 - K(\|X - y\|))\Big)\Big].$$

Using the properties of point processes and denoting the *adjusted attractiveness* by $\widetilde{a} := a + b$, we obtain

$$U^{\mathrm{crim}}(\lambda, s) = \int_W \widetilde{a}(x) e^{-\int_W K(\|x-y\|)\lambda(y)\mathrm{d}y} s(x)\mathrm{d}x - b. \tag{7}$$

We provide the derivation of this expression in Appendix B. Note that the criminal's problem is equivalent to one with no punishment, where the spatial attractiveness is replaced by the adjusted attractiveness. This happens because a criminal who successfully commits a crime earns b due to not being punished in addition to the crime profits a.

Formally, a potential criminal with an opportunity cost κ does not commit a crime if $\max_{s \in \mathcal{S}}\{U^{\mathrm{crim}}(\lambda, s)\} < \kappa$, and becomes a criminal if $\max_{s \in \mathcal{S}}\{U^{\mathrm{crim}}(\lambda, s)\} > \kappa$. When this last condition is satisfied, the support of the mixed strategy used by the criminal lies in the set $\mathrm{argmax}_{s \in \mathcal{S}}\{U^{\mathrm{crim}}(\lambda, s)\}$.

4.2 Equilibrium

We proceed to characterize the equilibrium. Formally, it is given by a police intensity $\lambda^\star$ and a criminal intensity $\mu^\star$, such that the three following conditions hold:

[13] For all results in this section, it is only required that the *average* number of potential criminals is N.

(i) The police command maximizes its expected utility, so $\int_W \lambda^\star(x)\mathrm{d}x \leq \Lambda$ and $U^{\mathrm{pol}}(\lambda^\star, \mu^\star) \geq U^{\mathrm{pol}}(\lambda, \mu^\star)$ for any function $\lambda \in \mathcal{C}_+$ with $\int_W \lambda(x)\mathrm{d}x \leq \Lambda$;

(ii) Each potential criminal with idiosyncratic cost κ commits a crime if, and only if, $\max_{s\in\mathcal{S}}\{U^{\mathrm{crim}}(\lambda^\star, s)\} > \kappa$ and each criminal has a location strategy in the set $\mathrm{argmax}_{s\in\mathcal{S}}\{U^{\mathrm{crim}}(\lambda^\star, s)\}$;

(iii) The individual and aggregate behavior of potential criminals is consistent–in particular, the expected number of criminals is $\int_W \mu^\star(x)\mathrm{d}x$.

Before analyzing the equilibrium behavior of each type of agent, we introduce some useful notation. First, we refer to the *police activity at* x as the expected sum of detection probabilities from all police patrols covering location x, given by $z^\star(x) := \int_W K(\|x-y\|)\lambda^\star(y)\mathrm{d}y$. Since $z^\star$ is an integral transform, Lerch's Theorem ensures a one-to-one relation between $z^\star$ and $\lambda^\star$.[14]

Second, we let $D(\cdot \,\|\, \cdot)$ be the Kullback-Leibler divergence

$$D(g \,\|\, h) := \int_W g(x) \ln \frac{g(x)}{h(x)} \mathrm{d}x,$$

for normalized densities g and h with support in W. This operator is also known as the relative entropy between g and h. We denote by u the uniform distribution over W and by U a uniform random variable.

Third, let $q := \int_W K(\|x\|)\mathrm{d}x$ be the effective coverage of each police patrol, a measure of the technology quality to detect crimes.

Additionally, for any function $f \in R_+^W$, define the constant

$$\underline{\Lambda}(f) := \frac{|W|}{q} \times \Big(\frac{\int_W \ln(f(x))\mathrm{d}x}{|W|} - \ln\min\{f\}\Big).$$

To simplify the exposition, we assume that the total available resources for patrolling respects $\Lambda > \underline{\Lambda}(\widetilde{a})$. This ensures that the police fully cover the entire region W in equilibrium.[15] If this condition is not met, the police covers only a subset of W and all

[14] Specifically, if another intensity measure $\bar{\lambda}^\star$ satisfies $\int_W K(\|x-y\|)\bar{\lambda}^\star(y)\mathrm{d}y = z^\star(x)$, then $\int_A(\lambda - \bar{\lambda})\mathrm{d}y = 0$ for each $A \subset W$. Thus, $\lambda^\star$ and $\bar{\lambda}^\star$ are identical intensity measures.

[15] The first term of the product is the inverse of the effective coverage proportion of each patrol, defined as $q/|W|$. A low ratio induces the police command to focus only on more attractive regions. The second term of the product is the difference between the average and the minimum of $\ln(\widetilde{a})$. When this term is large, the relative attractiveness of the region around $\mathrm{argmin}\{\widetilde{a}\}$ is small, and the police command might choose to turn a blind eye to the region.

our results hold in this subset.[16]

Lastly, we define the ACR in W as in the discrete-choice model, defined as the expected ratio of the number of crimes cleared and the number of crimes in equilibrium. Thus, denoting this average as $p^\star_W$ (the same notation used previously), we have

$$p^\star_W = \mathbb{E}\left[\frac{\sum_{x\in\Phi}\left(1-\prod_{y\in\Psi}\left(1-K(\|x-y\|)\right)\right)}{\sum_{x\in\Phi}\mathbb{1}_{\{x\in W\}}}\right].$$

This quantity can be seen as a measure of the police effectiveness.

The Police Command's Side

The first result of this section establishes a necessary condition for the police's optimal response, given the equilibrium crime intensity, corresponding to condition (i) of the equilibrium.

Lemma 1. *Suppose that $\Lambda > \underline{\Lambda}(\mu^\star)$ and that there is an equilibrium. Then, the police activity at x is increasing in the crime intensity $\mu^\star(x)$. More specifically, $\lambda^\star$ satisfies*

$$\forall x \in W, \quad z^\star(x) = \ln \mu^\star(x) + \frac{\Lambda q}{|W|} - \frac{\int_W \ln \mu^\star(x')\,dx'}{|W|}. \tag{8}$$

This lemma holds for any crime intensity, not necessarily the equilibrium one. From Equation (6), we already know that the police command tends to allocate more resources to regions where $\mu^\star$ is high. Equation (8) further specifies how: the optimal police activity at location x is an affine function of $\ln(\mu^\star)$.

The intuition behind this result is straightforward. The probability of clearing a crime is a concave function of police activity due to inefficiencies arising from overlapping patrol coverage. As a result, the optimal allocation of patrols is achieved when the marginal utility gain–i.e., the expected number of cleared crimes–is equalized across all points in W.

The Criminals' Side

Let us turn to the criminal's side of the equilibrium. We show that potential criminals adopt a cutoff strategy to decide whether to commit a crime, and we derive

[16] This subset is $\{x \in W : \widetilde{a}(x) \geq \overline{a}\}$ for a well-defined $\overline{a}$.

necessary conditions for their location strategies and the total number of criminals–corresponding to conditions (ii) and (iii) of the equilibrium.

Equation (7) implies that, if there is an equilibrium, a criminal with an opportunity cost κ commits a crime whenever

$$\max_{s\in\mathcal{S}}\left\{\int_W \widetilde{a}(x)e^{-z^\star(x)}s(x)\mathrm{d}x\right\} > \kappa + b.$$

The term $e^{-z^\star(x)} \times \widetilde{a}(x)$ is the product of the probability of committing a successful crime and the adjusted attractiveness at x. This term minus b is the expected payoff from committing a crime at x.

Since the $\int_W \widetilde{a}(x)e^{-z^\star(x)}s(x)\mathrm{d}x - b$ represents the mean of these benefits under the location strategy s (i.e., the distribution over W), any strategy with support in the set $\operatorname{argmax}_{x\in W}\{e^{-z^\star(x)}\widetilde{a}(x)\}$ can be the optimal response of a criminal to a patrolling intensity $\lambda^\star$. This argmax set is non-empty, provided that $\widetilde{a}$ and $\lambda^\star$ are continuous. Thus, if $\hat{x}$ is an element of this set, for any optimal location strategy,

$$\max_{s\in\mathcal{S}}\left\{\int_W \widetilde{a}(x)e^{-z^\star(x)}s(x)\mathrm{d}x\right\} = \widetilde{a}(\hat{x})e^{-z^\star(\hat{x})}.$$

This implies that a criminal commits a crime if $\kappa < \widetilde{a}(\hat{x})e^{-z^\star(\hat{x})} - b$. That is, a potential criminal commits a crime if and only if their opportunity cost is below a certain cutoff.

Moreover, because the cost of committing a crime is independent across individuals, the probability that a randomly chosen criminal commits a crime is $F_\kappa(\widetilde{a}(\hat{x})e^{-z^\star(\hat{x})} - b)$. Recalling that the number of potential criminals is N, the expected number of crimes is $NF_\kappa(\widetilde{a}(\hat{x})e^{-z^\star(\hat{x})} - b)$.

Characterization of the Equilibrium Intensities

We combine conditions (i)-(iii) of the equilibrium to prove the existence of an equilibrium and to characterize the crime rate, the patrolling and crime intensities, as well as the ACR.

Proposition 2. *Suppose that $\Lambda > \underline{\Lambda}(\widetilde{a})$. Then there exists an equilibrium and, in any equilibrium, the crime intensity $\mu^\star(x)$ is increasing in the adjusted attractiveness $\widetilde{a}(x)$. In addition, the equilibrium police activity $z^\star(x)$ depends only on $\widetilde{a}(x)$ and is increasing* *in it. More specifically, in any equilibrium, the value of committing a crime is given by*

$$V^\star := e^{-\frac{\Lambda q}{|W|}} \times e^{-D(u||f_{\widetilde{a}})} \times \int_W \frac{\widetilde{a}(x)}{|W|} dx - b, \tag{9}$$

the crime rate is $M^\star := NF_\kappa(V^\star)$*, the crime intensity is given by*

$$\mu^\star(x) = M^\star \frac{\widetilde{a}(x)}{\int_W \widetilde{a}(x')\,dx'} = M^\star f_{\widetilde{a}}(x), \quad \forall x \in W \tag{10}$$

and the police activity is

$$z^\star(x) = \ln f_{\widetilde{a}}(x) + \frac{\Lambda q}{|W|} + D(u||f_{\widetilde{a}}), \quad \forall x \in W\cdot \tag{11}$$

Moreover, the aggregate clearance rate in W *is*

$$p_W^\star = 1 - e^{-\frac{\Lambda q}{|W|}} \times e^{-D(u\,\|\,f_{\widetilde{a}})}. \tag{12}$$

We explain in detail each of the four results of the proposition.

The patrolling and crime intensities. The key to understanding the equilibrium police activity in Equation (11) is to note that the police must make criminals indifferent to the location of crime within W (as Equation (11) implies). Consider an alternative police intensity that, compared to $\lambda^\star$, implies a smaller criminal payoff (and thus a greater police intensity) in a given region A, and a greater criminal payoff in $W \setminus A$. Since criminals have more profitable options in $W \setminus A$, no crime would be committed in A, making it inefficient to allocate resources there. Therefore, the police would prefer to allocate its units more intensively in $W \setminus A$, contradicting the assumption that the police intensity is high in A. Hence, the police allocates patrols based solely on the adjusted attractiveness.

The behavior of criminals is described by Equation (10). In equilibrium, criminals commit crimes such that the marginal gain from increasing the police activity at each point is equalized. This may seem puzzling because criminal behavior is decentralized, but it is easy to understand if we interpret the Nash equilibrium as the stationary state of a repeated game. Suppose that, at some point in time, the crime intensity does not coincide with the equilibrium one. The police enforcement would start by moving toward wherever the crime intensity is higher than the equilibrium one–i.e., where the marginal benefit of increasing the police protection is higher. This would push criminals towards locations where police protection is lower than the equilibrium level. This process continues until the police returns to the point where criminals are indifferent to the crime location (the equilibrium crime intensity).

Finally, although in equilibrium crime and patrolling intensities are unique, the strategies of criminals are not unique, and it is only required that they choose crime strategies that generate the aggregate intensity $\mu^\star$. Indeed, our characterization holds even if criminals are assumed to coordinate.

The aggregate clearance rate. The ACR can be written as one minus the product of two factors, each capturing the fundamentals of criminals and the police. An improved clearance technology (i.e., a larger q) increases the proportion of cleared crimes through the technology term $e^{-\frac{\Lambda q}{|W|}}$. In addition, $p_W^\star$ is increasing in the number of patrols per unit area $\Lambda/|W|$, so a larger city requires more patrols to maintain the same ACR.

To better understand the meaning of the factor $e^{-D(u \,\|\, f_{\widetilde{a}})}$, we recall some properties of the Kullback-Leibler divergence. We have $D(u \,\|\, f_{\widetilde{a}}) \geq 0$ with equality if and only if $\widetilde{a}$ is constant. In addition, the 'closer' $\widetilde{a}$ is to being constant, the smaller the Kullback-Leibler divergence $D(u \,\|\, f_{\widetilde{a}})$. Therefore, $D(u \,\|\, f_{\widetilde{a}})$, the relative entropy between u and $f_{\widetilde{a}}$, can be informally seen as the distance of $f_{\widetilde{a}}$ to the uniform distribution and we use $D(u \,\|\, \cdot\,)$ as a measure of concentration.

The factor $\exp(-D(u \,\|\, f_{\widetilde{a}}))$ can be interpreted from an information-theoretic point of view. Criminal attractiveness distributed homogeneously over W does not reveal much information on the crime locations, thereby reducing police effectiveness. In particular, $p_W^\star$ is minimal when $\widetilde{a}$ is constant. Conversely, highly concentrated criminal activity conveys more information more information, increasing the police effectiveness.

The functional form we find for the ACR can be written as $1-k_1 e^{-k_2\Lambda}$, a continuous, differentiable, increasing, and concave function of Λ. If we assume that Λ is proportional to police expenditures on enforcement, this expression can be used to model public security technology in theoretical or structural models of crime. Besides, setting $\Lambda^*(W') := \int_{W'} \lambda(x)\mathrm{d}x$ (the police expenditures in W'), the ACR in each set $W' \subset W$ is given by (5), implying that our model is consistent. We thus provide a fully model-based justification for the several properties the literature imposes on the relation between the apprehension probability and the police expenditures (Ehrlich 1973, İmrohoroğlu et al. 2004, Fu and Wolpin 2017).

Finally, we observe that the ACR in Eq. (12)–which plays the same role as the PSPF by linking police expenditures to the clearance probability–is given by the functional form in Eq. (5). This implies that the ACR we obtain satisfies coherent aggregation. In other words, if the city is partitioned into cells and the equilibrium is solved within each cell using our model, the results can be aggregated to obtain citywide outcomes that do not depend on the particular partition used.

The equilibrium value of committing a crime and the crime rate. The crime rate increasing with the value of committing crime $V^\star$. From Equation (9), $V^\star$ is the product of three factors (minus the punishment). The first two factors also appear in the expression of the ACR: a denser (through $\Lambda/|W|$), better equipped (through q), and better informed (through $D(u \,\|\, f_{\widetilde{a}})$) police force more effectively deters criminals. The third factor is the most intuitive: the higher the average value of lootable goods per unit area, the greater the incentive to commit crimes.

Two main channels through which economic conditions impact crime were uncovered in Dube and Vargas (2013): the opportunity cost effect and the rapacity effect. These operate in our framework as follows. The opportunity cost is captured by the CDF F_κ. Specifically, a decrease in opportunity cost (for example, if potential criminals become poorer) increases the crime level, as more agents have a cost of engaging in crime below $V^\star$. The rapacity effect operates through $\int_W \widetilde{a}(x)/|W|, \mathrm{d}x$. When economic conditions improve only for the wealthiest, the total amount of lootable goods increases, while the outside option for individuals at risk of committing crimes remains unchanged. Since $V^\star$ rises and F_κ remains the same for relevant values, our framework predicts higher crime levels when income inequality increases.

Finally, $M^\star$ represents the minimum level of crime that any police deployment can achieve, as formalized in the following corollary.

Corollary 3. *The police activity given by $z^\star$ is the only one minimizing the crime rate.*

Since $z^\star$ is the unique police activity attaining this minimum, the police command problem can be interpreted as choosing deployments to minimize crime. The intuition is straightforward. Activity $z^\star$ equalizes the expected payoff from crime across all locations in W at the level $V^\star$. Any alternative activity $\widehat{z} \neq z^\star$ satisfying the same budget constraint must reallocate police resources across locations, decreasing criminals' payoffs at some points and increasing them at others. Because criminal activity concentrates where payoffs are highest, crime occurs at locations where the payoff exceeds $V^\star$. Thus, the maximal payoff under $\widehat{z}$ exceeds $V^{\star}$, inducing more agents to commit crimes and increasing aggregate crime.

4.3 A Numerical Example

This section presents a numerical example contrasting police deployment in the discrete and continuous models. As a benchmark, it can be shown that discrete models predict the number of police units per area (the analogue of police intensity in continuous models) to be proportional to the logarithm of crime attractiveness.

To simplify the exposition, we adapt our previous example to one-dimensional cities defined on the interval $[-1, 1]$. This allows us to depict crime attractiveness and police deployment in the same graph, thereby making a direct comparison more transparent. Patrol units positioned at x can detect crimes committed within $[x-r, x+r]$ with probability 1 (i.e., we use a uniform crime-detecting kernel), and we analyze cases in which r goes to zero (purple curves) and in which r is small but does not go to zero (orange curves). Since our objective is to illustrate how discrete models distort equilibrium police deployment by ignoring spillovers, we normalize the equilibrium responses so that the areas under the police intensity curve and the logarithm of crime attractiveness curve are equal.[17]

First, we consider City A, which has three peaks of crime attractiveness, as illustrated in panel (a) of Figure 5. When r goes to zero, police deployment closely matches the logarithm of crime attractiveness, coinciding with the equilibrium predicted by a consistent discrete model with sufficiently small cells. This similarity is expected: the near absence of spillover effects allows the continuous model to be well approximated by the discrete model when the spatial partition is sufficiently granular.

When r is not close to zero, this is no longer the case. The orange curve shows the equilibrium police deployment for $r = 0.35$. With a wider coverage radius, the equilibrium allocation increases police presence near the center of the city (around the first peak at $x = 0$), effectively shifting the distribution toward the city center. Police intensity at the two peripheral peaks (at $-x_0$ and x_0) is redistributed toward the center.

While it may initially seem that shifting police patrols toward the city center is optimal for wider coverage, this is not always the case. Panel (b) of Figure 5 shows the equilibrium police deployment for City B, which has two peaks of attractiveness. When

[17] Therefore, the solution assumes a large Λ that generates positive police allocation across the city. For any Λ large enough, the normalized police allocation would present the same distribution as the one depicted in the graph.

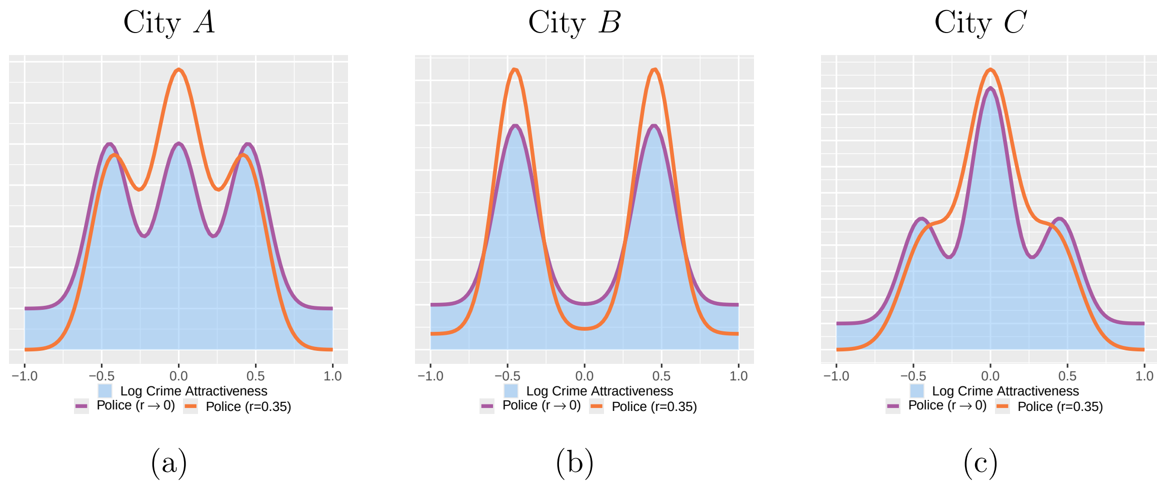


Figure 5: Each panel shows police deployment (i.e., police intensity) on the vertical axis for $r = 0.0001$ and $r = 0.35$ across different cities, plotted as a function of location on the horizontal axis. In all cities, attractiveness is given by $\log(a(x)) = \beta_0(e^{-\beta_1(x-x_0)^2} + e^{-\beta_1(x+x_0)^2}) + \beta_2 e^{-\beta_3 x^2} + \text{cte}$. For all cities we have $\beta_0 = 30$, $\beta_1 = 4$, $x_0 = -0.45$, $\beta_3 = 30$, and cte $= 1$. The only difference across the three cities is in the value of β_2. For City A, $\beta_2 = 4$, for City B, $\beta_2 = 0$, and lastly for City C, $\beta_2 = 9$. Further details of this numerical exercise are provided in the Supplementary Online Material.

police units can patrol larger areas ($r = 0.35$), the optimal allocation moves away from the center, increasing intensity at the two peaks. Police intensity around the city center is reduced by half. This occurs because the high concentration of patrols around the peaks generates spillovers of police protection into the center of City B, which is already a low-crime area. Consequently, there is no need to allocate too many additional patrols to the center.

A more extreme example is shown in panel (c) of Figure 5. Here, we consider a three-peak distribution, as in City A, but with a central peak of crime attractiveness higher than the peripheral peaks. When r goes to zero (purple curve), the equilibrium police deployment mirrors the three-peak distribution. However, as r increases (orange curve), the equilibrium allocation becomes unimodal. In this case, the high police intensity around the center already provides spillover protection to the peripheral peaks.

5 Conclusion

In this paper, we explore the advantages of using a topological rather than a discrete-choice model of crime to study spatial patterns of crime and police deployment. We begin by showing that, in existing spatial models of crime, equilibrium outcomes depend on how the city is partitioned. We then characterize a class of models–which we call consistent–whose equilibrium outcomes are invariant to the choice of spatial partition and that generate zero clearance probability in the absence of police protection.

We further argue that discrete-choice models (including consistent ones) cannot determine patrol locations: their reduced-form structure is problematic when cells are large, while the absence of an explicit topological structure precludes these models from incorporating spillovers. This is problematic when cells are small. To address this limitation, we propose a model of crime drawing on tools from the theory of point processes, endowed with an explicit topological structure to incorporate assumptions grounded in recent empirical evidence.

This model is tractable, allowing for a clearer understanding of the forces that drive spatial patterns of crime and police deployment and to derive policy implications. Moreover, it is highly flexible, enabling the incorporation of additional realistic features to study other phenomena. Although we focus on attractiveness as the sole source of spatial heterogeneity, the results remain analytical if we introduce heterogeneity along other dimensions: the reward for clearing crimes (e.g., the police command's payoff is lower if a crime is cleared in a poorer region), the punishment for crimes (e.g., harsher penalties for crimes committed in richer areas), and the clearance technology (e.g., regions with more streetlights or surveillance cameras). We leave these extensions for future work.

Our results caution against the functional forms commonly employed to relate police expenditures to clearance probability in structural models and propose a consistent relationship. Moreover, the paper proposes an empirical approach to uncover the spatial patterns of crime. By establishing an explicit relationship between observed crime and patrol locations and the unobserved attractiveness of each location, it becomes possible to cross-reference geographical data—such as population density, street lighting, or cost of living—with the inferred criminal attractiveness.

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Appendices

A Properties of PPPs Used in the Paper

We describe in the following a few other properties of PPPs that we use in this paper. Let Y be a PPP in W with intensity χ.

Intensity Defines the Distribution Law. The probability law of Y is entirely characterized by its intensity measure χ.

Characterization of the Process in Subsets. The set of points of Y lying in a set $A \subset W$ (i.e., the point process $A \cap Y$) inherits the Poisson properties, being a PPP with intensity $\chi(\cdot)\mathbb{1}_{\{(\cdot)\in A\}}$ and mean $\int_A \chi(x)\mathrm{d}x$.

The Thinning Property. We call *thinning property* the fact that if each point of Y is removed independently and with a probability p_{th} (that might depend on the location in W), the remaining set of points is a PPP with intensity $(1-p_{\text{th}})\chi$.

The Generating Functional. For any measurable function f, $\mathbb{E}\left[\prod_{x\in Y} f(x)\right]$, called *generating functional* with respect f, is equal to $\exp\left(-\int_W (1-f(x))\chi(x)\mathrm{d}x\right)$.

B Proofs

Proof of Proposition 1

Proof. We prove it in three steps.

First step: finding a general functional form for π. Since the crime rate in a subset of W must not depend on the way we partition it (condition (i) of a coherent aggregation), and the crime rate is a function of the value of crime, $V^\star$ must be independent of the partition $\{W_i\}_{i\in I}$.

To simplify notation, we let $\widetilde{a}(W') := a(W') + b$ for each measurable set $W' \subseteq W$. Let W_1, W_2 be disjoint subsets of W. Then, by condition (ii) of coherent aggregation, we must have $p^\star_{\{W_1,W_2\}} = p^\star_{\{W_1\cup W_2\}}$, so

$$1-(V^\star+b)\frac{\sum_{i=1}^2 \frac{\pi_{W_i}^{-1\prime}(1-(V^\star+b)/\widetilde{a}(W_i))}{\widetilde{a}(W_i)}}{\sum_{i=1}^2 \pi_{W_i}^{-1\prime}(1-(V^\star+b)/\widetilde{a}(W_i))} = 1-(V^\star+b)\frac{1}{\frac{1}{|W_1+W_2|}\int \widetilde{a}(x)\mathrm{d}x}.$$

Using the fact that W_1 and W_2 are disjoint and simplifying the above expression, we have

$$\frac{\sum_{i=1}^2 \frac{\pi_{W_i}^{-1\prime}(1-(V^\star+b)/\widetilde{a}(W_i))}{\widetilde{a}(W_i)}}{\sum_{i=1}^2 \pi_{W_i}^{-1\prime}(1-(V^\star+b)/\widetilde{a}(W_i))} = \frac{|W_1|+|W_2|}{|W_1|\widetilde{a}(W_1)+|W_2|\widetilde{a}(W_2)}.$$

Rearranging the expressions, we obtain that

$$\frac{\pi_{W_1}^{-1\prime}(1-(V^\star+b)/\widetilde{a}(W_1))}{|W_1|\widetilde{a}(W_1)} = \frac{\pi_{W_2}^{-1\prime}(1-(V^\star+b)/\widetilde{a}(W_2))}{|W_2|\widetilde{a}(W_2)}.$$

Since W_1 and W_2 are arbitrary, we must have that, for any $W' \subset W$

$$\frac{\pi_{W'}^{-1\prime}(1-(V^\star+b)/\widetilde{a}(W'))}{|W'|\widetilde{a}(W')} = C$$

for some constant C. Making $|W'|$ a constant, this is a differential equation. We do $u = 1 - (V^\star + b)/\widetilde{a}(W')$ to obtain

$$\pi_{W'}^{-1\prime}(u) = C\,|W'|\,\widetilde{a}(W')(u) = \frac{C\,|W'|\,(V^\star + b)}{1-u}.$$

Then we solve it to find

$$\pi_{W'}^{-1}(u) = -C\,|W'|\,(V^\star + b)\ln(1-u) + G(W'),$$

so

$$\pi_{W'}(\Lambda') = 1 - \exp\left(-\frac{\Lambda'}{C\,|W'|\,(V^\star + b)}\right)\exp\left(\frac{G(W')}{C\,|W'|\,(V^\star + b)}\right).$$

We let $C(V^\star + b) := 1/\theta$ and, since $G(W')$ can be any function that does not depend on Λ', we define

$$\exp(G(W')/(C|W'|(V^\star + b))) =: g(W'),$$

so

$$\pi_{W'}(\Lambda') = 1 - e^{-\frac{\Lambda'\theta}{|W'|}} g(W').$$

Second step: finding the family of functions defining g. To obtain the relation on g, we consider conditions (i) and (iii) of coherent aggregation, and we use this functional form in the utility of criminals. Consider a partition $\{W_i\}_{i\in I}$ of W and let Λ' be the police expenditures in $\bigcup_{i\in I} W_i$. From condition (i), for each i:

$$V^\star + b = \widetilde{a}(W_i)e^{-\frac{\Lambda_i\theta}{|W_i|}} g(W_i),$$

so

$$\Lambda_i = \frac{|W_i|}{\theta}\ln\left(\frac{\widetilde{a}(W_i)g(W_i)}{V^\star + b}\right).$$

From condition (iii) of coherent aggregation,

$$\sum_{i\in I}\frac{|W_i|}{\theta}\ln\left(\frac{\widetilde{a}(W_i)g(W_i)}{V^\star + b}\right) = \Lambda'.$$

This implies that $g(W') > \epsilon > 0$ for each $W' \in W$ (otherwise the left-hand term would be negative when I is a singleton) and that

$$\sum_{i\in I} |W_i| \ln(\widetilde{a}(W_i)g(W_i)) = \Big|\bigcup_{i'\in I} W_{i'}\Big| \ln\Big(\widetilde{a}\Big(\bigcup_{i\in I} W_i\Big) g\Big(\bigcup_{i\in I} W_i\Big)\Big).$$

Let $H(W') := |W'| \ln(\widetilde{a}(W')g(W'))$ for any $W' \in W$ and impose $H(\emptyset)$ to be equal to $\lim_{|W'|\to 0} H(W') = 0$. Then we have

$$\sum_{i\in I} H(W_i) = H\Big(\bigcup_{i\in I} W_i\Big).$$

That is, H is finite additive with $H(\emptyset) = 0$, so H is a σ-finite measure. Besides, $H(W') = 0$ when $|W'| = 0$ (recall that $\widetilde{a}(W')$ and $g(W')$ are both positive), so H is absolutely continuous with respect to Lebesgue measure. Hence, by the Radon–Nikodym theorem, there exists a unique integrable function (up to Lebesgue-a.e. equality) $h : W \to \mathbb{R}$ such that for all $W' \subset W$ we have $H(W') = \int_{W'} h(x)\mathrm{d}x$. Since $H(W') = |W'| \ln(\widetilde{a}(W')g(W'))$, we have that

$$e^{\frac{1}{|W'|}\int_{W'} h(x)\mathrm{d}x} = g(W')\widetilde{a}(W')$$

and isolating $g(W')$ we obtain the desired relation on g.

Third step: finding the unique g. We show the uniqueness of g under the NENC principle. Take an arbitrary sequence $\{W_n\}_n$ converging to an arbitrary point $x \in W$ and let $\lim_n \Lambda_n/|W_n| = 0$. Because the NENC principle holds,

$$0 = \lim_n \pi_{W_n}(\Lambda_n) = \lim_n \Big(1 - e^{\frac{-\Lambda_n \theta}{|W_n|}} g(W_n)\Big)$$

Due to the continuity of g and exp, we have $0 = 1 - g(\lim_n W_n)$ and using the representation of g as a function of h we have

$$1 = \lim_n \frac{e^{\frac{1}{|W_n|}\int_{W_n} h(x')\mathrm{d}x'}}{\widetilde{a}(W_n)} \Leftrightarrow \ln(\widetilde{a}(x)) = \lim_n \frac{1}{|W_n|}\int_{W_n} h(x')\mathrm{d}x'$$

Note that we need h to be continuous so the limit in the right-hand term is well defined at each point in W. In this case, we have $\lim_n \frac{1}{|W_n|}\int_{W_n} h(x')\mathrm{d}x' = \int_{\mathbb{R}^2} \delta(x'-x)h(x')\mathrm{d}x' = h(x)$. Thus, the only function respecting the NENC principle is $h = \ln \widetilde{a}$, which concludes the proof. □

Proof of Equation (6)

Using the independency between Φ and Ψ, we have that

$$U^{\mathrm{pol}}(\lambda,\mu) = \mathbb{E}\Big[\sum_{x\in\Phi}\Big(1-\mathbb{E}\Big[\prod_{y\in\Psi}(1-K(\|x-y\|))\,\Big|\Phi\Big]\Big)\Big].$$

Then, using generating functional property, we have

$$E\Big[\prod_{y\in\Psi}(1-K(\|x-y\|))\,\Big|\Phi\Big] = e^{-\int_W K(\|x-y\|)\lambda(y)\mathrm{d}y}$$

and thus

$$U^{\mathrm{pol}}(\lambda,\mu) = \mathbb{E}\Big[\sum_{x\in\Phi}\Big(1-e^{-\int_W K(\|x-y\|)\lambda(y)\mathrm{d}y}\Big)\Big].$$

Finally, we use the intensity property in the above expression to obtain the expression of $U^{\mathrm{pol}}(\lambda,\mu)$ as a function of μ and λ. □

Proof of Equation (7)

Since X and Ψ are independent, the payoff from committing a crime at X can be written as

$$\mathbb{E}\Big[a(X)\mathbb{E}\Big[\prod_{y\in\Psi}(1-K(\|X-y\|))\,\Big|X\Big]-b\Big(1-\mathbb{E}\Big[\prod_{y\in\Psi}(1-K(\|X-y\|))\,\Big|X\Big]\Big)\Big].$$

Using the generating functional property in this expression, the above expression becomes

$$\mathbb{E}\Big[a(X)e^{-\int_W K(\|X-y\|)\lambda(y)\mathrm{d}y}-b\left(1-e^{-\int_W K(\|X-y\|)\lambda(y)\mathrm{d}y}\right)\Big].$$

Taking the expectation and rearranging terms results in the desired criminal's utility expression. □

Proof of Lemma 1

Proof. Observe that $\Lambda > \underline{\Lambda}(\mu^\star)$ implies that $z^\star > 0$ for each $x \in W$ and the budget is respected, so the police activity described by Equation (8) is feasible. The police command's problem can be equivalently formulated as minimizing $\int_W \exp\big(-\int_W K(\|x-$

$y\|)\widetilde{\lambda}(y)\mathrm{d}y\big)\mu^\star(x)\mathrm{d}x$ with respect to $\widetilde{\lambda}$ positive, continuous in W, and such that $\int_W \widetilde{\lambda}(x)\mathrm{d}x = \Lambda$. Take an arbitrary λ in this set. By the Fubini's theorem and the symmetry of K, we have

$$\int_W \int_W K(\|x-y\|)\lambda(y)\mathrm{d}y\,\mathrm{d}x = \int_W \lambda(y)\int_W K(\|x-y\|)\mathrm{d}x\,\mathrm{d}y = \int_W \lambda(y)q\mathrm{d}y = \Lambda q.$$

Using the Jensen's inequality and the convexity of $z \mapsto \exp(-z)$, we have

$$\begin{aligned}
&\int_W \exp\Big(-\int_W K(\|x-y\|)\lambda(y)\mathrm{d}y\Big)\mu^\star(x)\,\mathrm{d}x \\
&\qquad = |W|\left(\frac{1}{|W|}\int_W \exp\Big(-\int_W K(\|x-y\|)\lambda(y)\mathrm{d}y + \ln(\mu^\star(x))\Big)\Big)\,\mathrm{d}x\right) \\
&\qquad \geq |W|\exp\left(-\frac{1}{|W|}\int_W \left(\int_W K(\|x-y\|)\lambda(y)\mathrm{d}y - \ln(\mu^\star(x))\right)\mathrm{d}x\right) \\
&\qquad = |W|\exp\left(-\frac{1}{|W|}\int_W\int_W K(\|x-y\|)\lambda(y)\mathrm{d}y\,\mathrm{d}x + \frac{1}{|W|}\int_W \ln(\mu^\star(x))\,\mathrm{d}x\right) \\
&\qquad = |W|\exp\left(-\frac{\Lambda q}{|W|} + \frac{1}{|W|}\int_W \ln(\mu^\star(x))\,\mathrm{d}x\right).
\end{aligned}$$

It is easy to check that any λ satisfying (8) attains the bound above. In addition, any λ that does not satisfy (8) does not attain the bound, proving that measures that satisfy (8) are the only ones attaining the minimum. □

Proof of Proposition 2

Proof. Observe that $\Lambda > \underline{\Lambda}(\widetilde{a})$ implies that, for each $x \in W$, $z^\star(x)$ given by (11) is positive. Moreover, the budget is respected, so the police activity described by Equation (11) is feasible. We first prove Equation (11), followed by Equation (9), Equation (10), and finally Equation (12).

Police activity. We show that $z^\star$ defined by Equation (11) is the only possible equilibrium police activity. In this part of the proof, $\mu^\star$ is the equilibrium crime intensity, but we do not impose that it respects Equation (10). Suppose, by way of contradiction, that there is an equilibrium police activity $\widehat{z}$ given by an intensity measure $\widehat{\lambda}$ that is different from $\lambda^\star$. By the Lerch's Theorem (i.e., the one-to-one relation between police activity and police intensity), $\widehat{z} \neq z^\star$. Observe that, for each $x \in R$

$$\widetilde{a}(x)e^{-z^\star(x)} = e^{-\frac{\Lambda q}{|W|}} e^{D(u||f_{\widetilde{a}})} \int_W \widetilde{a}(x')\mathrm{d}x' := c^\star.$$

That is, $\widetilde{a}e^{-z^\star}$ is constant. Since $\widehat{z} \neq z^\star$, $\widetilde{a}e^{-\widehat{z}}$ is not constant. Also, recall that we restrict $\widehat{\lambda}$ to be continuous, so $\widehat{z}$ is continuous and so it is $\widetilde{a}(x)e^{-\widehat{z}(x)}$. Therefore, $\widetilde{a}e^{-\widehat{z}}$ attains a maximum in W, which we denote as $\widehat{c}$. Because $\widetilde{a}e^{-\widehat{z}}$ is not constant, $\widehat{A} := \{x \in W : e^{-\widehat{z}(x)}\widetilde{a}(x) = \widehat{c}\}$ is a proper subset of W. Due to the maximizing behavior of criminals imposed by condition (ii) of the equilibrium, all crime must occur in $\widehat{A}$ (i.e., we would need $\mu^\star(x) = 0$ if $x \notin \widehat{A}$), leading to

$$U^{\mathrm{pol}}(\widehat{\lambda}, \mu^\star) = \int_W (1 - \widehat{c}/\widetilde{a}(x))\mu^\star(x)\mathrm{d}x.$$

We prove now that setting $z^\star$ would give the police command a greater payoff. Because $\widehat{z}$ and $z^\star$ are distinct but both represent feasible police activities, we have $\int_W z^\star(x)\mathrm{d}x = \int_W \widehat{z}(x)\mathrm{d}x$. Hence, there exist points in W where $z^\star(x) > \widehat{z}(x)$, and others where $z^\star(x) < \widehat{z}(x)$. This means that the set $B := \{x \in W : \widehat{z}(x) < z^\star(x)\}$ is non-empty. In addition, observe that $\widetilde{a}(x)e^{-z^\star(x)} < \widetilde{a}(x)e^{-\widehat{z}(x)}$ if, and only if, $\widehat{z}(x) < z^\star(x)$, thus, $B = \{x \in W : \widetilde{a}(x)e^{-z^\star(x)} < \widetilde{a}(x)e^{-\widehat{z}(x)}\}$. Recalling that $\widetilde{a}e^{-z^\star} = c^\star$, we have $B = \{x \in W : c^\star < \widetilde{a}(x)e^{-\widehat{z}(x)}\}$. Let $x' \in \widehat{A}$. Then, for any $x \in B$,

$$\widehat{c} = \widetilde{a}(x')e^{-\widehat{z}(x')} \geq \widetilde{a}(x)e^{-\widehat{z}(x)} > c^\star = \widetilde{a}(x)e^{-z^\star(x)},$$

implying that $\widehat{A} \subset B$. In particular, if $x' \in \widehat{A}$, then $(1 - e^{-z^\star(x')}) > 1 - \widehat{c}/\widetilde{a}(x')$, yielding

$$\begin{aligned} U^{\mathrm{pol}}(\widehat{\lambda}, \mu^\star) &= \int_W (1 - \widehat{c}/\widetilde{a}(x))\mu^\star(x)\mathrm{d}x = \int_{\widehat{A}} (1 - \widehat{c}/\widetilde{a}(x))\mu^\star(x)\mathrm{d}x \\ &< \int_{\widehat{A}} (1 - e^{-z^\star(x)})\mu^\star(x)\mathrm{d}x = \int_W (1 - e^{-z^\star(x)})\mu^\star(x)\mathrm{d}x = U^{\mathrm{pol}}(\lambda^\star, \mu^\star), \end{aligned}$$

so $\widehat{\lambda}$ does not maximize the police command's payoff, and condition (i) of the equilibrium is not respected. Thus, no police activity different from $z^\star$ can be the equilibrium intensity.

Expected gain from committing a crime. To show Equation (9), it suffices to replace $z^\star$ defined by Equation (11) in the value of committing crimes, given by $\widetilde{a}(x)e^{-z^\star(x)}$

for any $x \in W$.

Crime intensity. Now, we show that $\mu^\star$ defined by Equation (10) is the only possible equilibrium crime intensity. The police activity defined by Equation (11) must be the best response for the crime intensity so that condition (i) of the equilibrium is respected. But the best response for the crime intensity is characterized by Equation (8). Hence, we must have

$$\forall x \in W, \quad \ln \mu^\star(x) + \frac{\Lambda q}{|W|} - \frac{\int_W \ln \mu^\star(x')\mathrm{d}x'}{|W|} = \ln f_{\widetilde{a}}(x) + \frac{\Lambda q}{|W|} + D(u||f_{\widetilde{a}}).$$

Rewriting this relation, if there is $\mu^\star$ solving this functional equation, it respects

$$\forall x \in W, \quad \ln\left(\frac{\mu^\star(x)}{\widetilde{a}(x)}\right) = \ln(c)$$

for a positive constant c, implying that $\mu^\star = c \times \widetilde{a}$. However due to condition (iii) of equilibrium, $\int_W \mu^\star(x)\mathrm{d}x = \int_W c\widetilde{a}(x)\mathrm{d}x = M^\star$, so the only value of c our equilibrium admits is $c = M^\star/(\int_W \widetilde{a}(x)\mathrm{d}x)$.

ACR. Because Φ and Ψ are independent, and using the generating functional of a PPP, we have that

$$\begin{aligned}
p_W^\star &= \mathbb{E}\left[\frac{\sum_{x\in\Phi}\Big(1 - \prod_{y\in\Psi}\left(1 - K(\|x-y\|)\right)\Big)}{\sum_{x\in\Phi}\mathbb{1}_{\{x\in W\}}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\frac{\sum_{x\in\Phi}\Big(1 - \prod_{y\in\Psi}\left(1 - K(\|x-y\|)\right)\Big)}{\sum_{x\in\Phi}\mathbb{1}_{\{x\in W\}}}\middle|\Phi\right]\right] \\
&= \mathbb{E}\left[\frac{\sum_{x\in\Phi}(1 - e^{-z^\star(x)})}{\sum_{x\in\Phi}\mathbb{1}_{\{x\in W\}}}\right] \\
&= \sum_{m=0}^{\infty}\Pr\Big(\sum_{x\in\Phi}\mathbb{1}_{\{x\in W\}} = m\Big)\mathbb{E}\left[\frac{\sum_{x\in\Phi}(1 - e^{-z^\star(x)})}{\sum_{x\in\Phi}\mathbb{1}_{\{x\in W\}}}\middle|\sum_{x\in\Phi}\mathbb{1}_{\{x\in W\}} = m\right]
\end{aligned}$$

If $\{x_1, x_2, \ldots, x_m\}$ represents Φ when the number of points is m, the expectation above can be written as

$$\mathbb{E}\left[\frac{\sum_{j=1}^{m}(1-e^{-z^\star(x_j)})}{m}\right] = \frac{1}{m}\sum_{j=1}^{m}\mathbb{E}\left[(1-e^{-z^\star(x_j)})\right] = \mathbb{E}\left[(1-e^{-z^\star(x)})\right].$$

Since the average distribution of a given crime is $\mu^\star/M^\star$,

$$\begin{aligned}
p_W^\star &= \sum_{m=0}^{\infty}\Pr\Big(\sum_{x\in\Phi}\mathbb{1}_{\{x\in W\}} = m\Big)\mathbb{E}\left[(1-e^{-z^\star(x)})\right] \\
&= E\left[(1-e^{-z^\star(x)})\right] \\
&= \int_W (1-e^{-z^\star(x)})\frac{\mu^\star(x)}{M^\star}\mathrm{d}x \\
&= 1 - \int_W e^{-z^\star(x)} f_{\widetilde{a}}(x)\mathrm{d}x.
\end{aligned}$$

To conclude the proof, we replace the value of $z^\star$ given by Equation (11) in the above expression and rearrange it to obtain equation (12). □

Proof of Corollary 3

Proof. Consider the notation used to obtain the equilibrium police activity $z^\star$ in the proof of Proposition 2. Under the arbitrary police activity $\widehat{z} \neq z^\star$, all crime is committed in B. Define $\widehat{V}$ as the payoff of criminals for committing a crime in B, we have

$$\widehat{V} = \widetilde{a}(x)e^{-\widehat{z}(x)} - b > c^\star - b = V^\star.$$

Therefore, the number of criminals under $\widehat{z}$ is given by $NF_\kappa(\widehat{V}) > NF_\kappa(V^\star) = M^\star$. □

Supplementary Online Appendix

From Police Activity to Police intensity

In Subsection 4.2, we defined the relationship between police intensity (the density of patrols at each point) and police activity (the level of protection generated by patrol locations) as

$$z^\star(x) := \int_W K(\|x-y\|)\lambda^\star(y)\,\mathrm{d}y$$

for each $x \in W$. This implies that $z^\star$ is an integral transform of $\lambda^\star$ (similar to the Laplace or Fourier transforms). Lerch's theorem then implies that there is a one-to-one correspondence between them.

It is straightforward to obtain $z^\star$ when $\lambda^\star$ is known. However, to determine the equilibrium police locations, we must recover the inverse relation, since Proposition 2 characterizes $z^\star$ rather than $\lambda^\star$. This appendix explains how to recover $\lambda^\star$ in order to generate the numerical examples in Subsection 4.3. Recall that we use the uniform kernel with $K = \mathbb{1}_{\{(\cdot)\le r\}}$ and that, to obtain analytical expressions, we ignore border effects and assume that r is small relative to the size of W.

The reasoning proceeds as follows. Suppose that $\lambda^\star(x) = 1$ for all x. In this case, we have $z^\star(x) = \int_{x-r}^{x+r} 1\,\mathrm{d}t = 2r$. Since the relationship between $z^\star$ and $\lambda^\star$ is one-to-one, it follows that if $z^\star(x) = c_0$ for each x, then $\lambda^\star(x) = c_0/(2r)$ for each x. It is straightforward to generalize this result and show that if either $\lambda^\star$ or $z^\star$ is a polynomial of order n, then the other is also a polynomial of order n.

To exemplify it, suppose now that $\lambda^\star$ is a polynomial of degree 3, and note that

$$\begin{aligned}
\int_{x-r}^{x+r} 1\mathrm{d}t &= 2r,\\
\int_{x-r}^{x+r} t\mathrm{d}t &= 2rx,\\
\int_{x-r}^{x+r} t^2\mathrm{d}t &= 2rx^2 + \frac{2}{3}r^3,\\
\int_{x-r}^{x+r} t^3\mathrm{d}t &= 2rx^3 + 2r^3x.
\end{aligned}$$

Define

$$\begin{bmatrix} 2r & 0 & \frac{2}{3}r^3 & 0 \\ 0 & 2r & 0 & 2r^3 \\ 0 & 0 & 2r & 0 \\ 0 & 0 & 0 & 2r \end{bmatrix} = 2r \begin{bmatrix} 1 & 0 & \frac{1}{3}r^2 & 0 \\ 0 & 1 & 0 & r^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =: 2rA_3$$

Then, using the base $\{1, x, x^2, x^3\}$, we can define a linear transformation from the space of the police locations to the space of the expected number of patrols detecting a crime such that

$$\int_{x-r}^{x+r} \lambda^\star(t)\mathrm{d}t = z^\star(x).$$

If $\lambda^\star(t) = c_3t^3 + c_2t^2 + c_1t + c_0$, we represent it as $\lambda^\star = [c_0\ c_1\ c_2\ c_3]'$ and if $z^\star(x) = d_3x^3 + d_2x^2 + d_1x + d_0$ we represent it by $z^\star = [d_0\ d_1\ d_2\ d_3]'$. Thus

$$2rA_3\lambda^\star = z^\star \Rightarrow \lambda^\star = (1/2r)A_3^{-1}g^\star$$

where

$$A_3^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{3}r^2 & 0 \\ 0 & 1 & 0 & -r^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This means that we have an *exact* expression for $\lambda^\star$ as long as $z^\star$ is at most a cubic function. In fact, $A_n = \{a_{ij}\}$ has a general expression that is simple to calculate:

$$a_{ij} = \frac{(j-1)!}{(j-i+1)!(i-1)!}r^{j-i}$$

if $j \geq i$ and $j + i$ is even, and $a_{ij} = 0$ otherwise (we don't index a_{ij} on n since it does not depend on n).

Even if $z^\star$ is not a polynomial but an analytic function (so that its Taylor expansion is well defined), we can use this technique to obtain $\lambda^\star$ to any desired level of precision. Let a_{ij}^{-1} denote the (i, j)-th entry of A_n^{-1}. We know that A_n^{-1} is upper triangular with ones on the diagonal (for being triangular, A_n is easy to invert). We can therefore compute the explicit expression of $\lambda^\star$ in an infinite basis. Let $z^\star(x) = \sum_{j=0}^{\infty} d_j x^j$, then

$$c_0 = \sum_{j=0}^{\infty} a^{-1}_{1j+1} d_j, \quad c_1 = \sum_{j=1}^{\infty} a^{-1}_{2j+1} d_j,$$

and so on, so

$$c_i = \sum_{j=i}^{\infty} a^{-1}_{i+1j+1} d_j.$$

Therefore, $\lambda^\star$ is given by

$$\lambda^\star(t) = \frac{1}{2r} \sum_{i=0}^{\infty} t^i \Big(\sum_{j=i}^{\infty} a^{-1}_{i+1j+1} d_j \Big).$$

The approximate expression of $\lambda^\star$ is given by

$$\lambda^\star(t) = \frac{1}{2r} \sum_{i=0}^{n} t^i \Big(\sum_{j=i}^{n} a^{-1}_{i+1j+1} d_j \Big),$$

where n can be chosen sufficiently large to ensure a good approximation. In our numerical example, we set $n = 170$.